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An analysis on the thermal instability of forced convection flow over isothermal horizontal flat plate

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Abstract—Thermal instability of forced convection flow over an isothermal horizontal flat plate in the form of longitudinal vortices is examined by introducing three-dimensional spatial dependence of the perturbation quantities. The system of stability equations has been simplified significantly by considering the limiting case of very large Prandtl numbers and by seeking similarity solutions for the amplitude functions of the perturbation quantities. The effect of x-dependent temperature perturbation is shown to stabilize the flow as compared with x-independent temperature perturbation, which explains very well the extant theoretical results and experimental observations.

1. INTRODUCTION

THE OCCURRENCE of longitudinal vortex rolls in a laminar flow over an isothermal horizontal flat plate has drawn the attention of many researchers since Wu and Cheng [1] first reported their result of linear stability analysis. In most studies performed thereafter [2], it has been customary to consider that the stationary and neutrally stable perturbation quantities are independent of the x (main flow direction) coordinate. This idea originated from the excellent discussion by Haaland and Sparrow [3] who investigated the onset of vortex instability in natural convection flow adjacent to a heated inclined flat plate. Such a choice of perturbation quantities may have to be re-examined when purely forced (or mixed) convection basic flow is employed, as suggested recently by Chen and Chen [4], who performed a meticulous numerical analysis for the thermal instability of the family of Falkner–Skan flows. Therefore, in this study, we are mainly interested in how the x dependence of perturbation quantities will affect the onset criterion of thermal instability of forced convection flow.

In order to examine the validity of this viewpoint preferentially, the system of stability equations is simplified by restricting it to the case of very large Prandtl numbers. This simplification is made on the basis of the observation that as the Prandtl number becomes large the critical Grashof number and the critical wave number depend on Prandtl number only weakly. For example, Hwang and Cheng [5] who analysed the thermal instability of laminar natural convection flow on inclined isothermal plates found that the critical Rayleigh number is a weak function of Prandtl number. Choi [6] and Davis and Choi [7] also presented, in their investigations on the onset of cellular convection in a flowing liquid layer, that there is a good agreement between the theoretical result obtained with infinite Prandtl number and the experimental result using water ($Pr = 7$).

The partial differential equations for the amplitude functions of perturbation quantities are further reduced to ordinary differential equations by introducing a similarity variable which is based on the thermal boundary layer thickness. Then, the non-parallel and the parallel basic flow models are represented in the form of approximating polynomials which are usually adopted in the integral method of a forced convection boundary layer problem so that the perturbation quantities may be sought in the convenient form of a fast convergent power series. The difficulty commonly encountered in representing the boundary layer conditions in the main stream region outside the boundary layer is overcome by replacing it with the condition at the edge of the thermal boundary layer or by adopting the assumption of the so-called bottling effect of the temperature perturbation [3, 7]. The advantage of this approach is that the critical values marking the onset of thermal instability can be obtained simply as the solution of a 6×6 (or 5×5) determinant which consists of the conditions at the interface between the thermal boundary layer and the outer region only.

2. ANALYSIS

We consider the vortex instability of a steady laminar boundary layer flow over an isothermal horizontal flat plate, which is maintained at a constant temperature T_w. The free stream velocity is U_∞ and the free stream temperature T_∞. The fluid properties are assumed to be constant except that the density variations are considered only to the extent that they

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NOMENCLATURE

Gr_x	local Grashof number, $g\beta\Delta t\, x^3/\nu^2$
g, $\mathbf{g}$	gravitational acceleration
L_0	a constant characteristic length
P_b, p, p'	basic pressure, pressure, pressure perturbation
Pr	Prandtl number κ/ν
Ra	Rayleigh number, $g\beta\Delta t\,\delta_t^3/\kappa\nu$
Ra_0	Rayleigh number, $g\beta\Delta t(L_0\,Re^{-1/2}\,Pr^{-1/3})^3/\kappa\nu$
Re	Reynolds number, $U_\infty L_0/\nu$
Re_x	local Reynolds number, $U_\infty x/\nu$
T_b	basic temperature
T_w, T_∞	wall temperature, free stream temperature
t, t'	temperature, temperature perturbation
Δt	temperature difference, $T_w - T_\infty$
(U_b, W_b)	basic velocity components
U_∞	free stream velocity
$\mathbf{u}$	velocity vector
(u, v, w)	velocity components
(u', v', w')	components of velocity perturbation
(x, y, z)	rectangular Cartesian coordinates (see Fig. 1).

Greek symbols

α	wave number
β	thermal expansion coefficient
δ_h, δ_t	velocity boundary layer thickness, thermal boundary layer thickness
ζ	similarity variable, $z/\delta_t(x)$
κ	thermal diffusivity
ν	kinematic viscosity
ρ	density.

Superscripts

$'$	perturbation quantities
$\bar{}$	dimensionless variables based on L_0
$\tilde{}$	dimensionless variables based on $\delta_t(x)$
$*$	amplitude functions of perturbation quantities.

contribute to the buoyancy forces. Thus, the starting point for the present problem is the Boussinesq form of the conservation laws

$$\nabla\cdot\mathbf{u} = 0 \tag{1}$$

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p - \mathbf{g}\beta(t - T_\infty) + \nu\nabla^2\mathbf{u} \tag{2}$$

$$\mathbf{u}\cdot\nabla t = \kappa\nabla^2 t \tag{3}$$

where the coordinate system is taken as schematically shown in Fig. 1.

As discussed in refs. [1, 8], the basic flow is a two-dimensional forced or mixed convection boundary layer flow which depends on x and z, and is denoted by U_b, $V_b = 0$, W_b, P_b and T_b. According to linear stability theory, the infinitesimal perturbation quantities are superposed on the basic quantities. Based on the arguments given in the first part of the Introduction, we assume that the derivatives of the perturbation quantities with respect to x may not be zero. Hence, the perturbed flow can be represented as

$$\begin{aligned} u &= U_b(x,z) + u'(x,y,z) \\ v &= v'(x,y,z) \\ w &= W_b(x,z) + w'(x,y,z) \\ t &= T_b(x,z) + t'(x,y,z) \\ p &= P_b(x,z) + p'(x,y,z). \end{aligned}$$

Substituting the above expressions into equations (1)–(3), subtracting the parts satisfied by the basic quantities and linearizing the remaining parts, we obtain the stability equations. In non-dimensionalizing these equations, we follow Chen and Chen's [4] argument by introducing the constant length scales, L_0 and $L_0\,Re^{-1/2}\,Pr^{-1/3}$, respectively, for streamwise and normal variations, where the thermal boundary layer thickness δ_t is known to vary approximately as the latter. Using these length scales, the dimensionless variables are

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{L_0}(x, y\,Re^{1/2}\,Pr^{1/3}, z\,Re^{1/2}\,Pr^{1/3})$$

$$(\bar{U}_b, \bar{W}_b) = \frac{1}{U_\infty}(U_b\,Pr^{1/3}, W_b\,Re^{1/2}\,Pr^{2/3})$$

$$(\bar{u}, \bar{v}, \bar{w}) = \frac{1}{U_\infty}(u'\,Pr^{1/3}, v'\,Re^{1/2}\,Pr^{2/3}, w'\,Re^{1/2}\,Pr^{2/3})$$

$$\bar{\theta}_b = \frac{T_b - T_\infty}{\Delta t}$$

$$\bar{p} = \frac{p'}{\rho U_\infty^2}\,Re\,Pr^{1/3}.$$

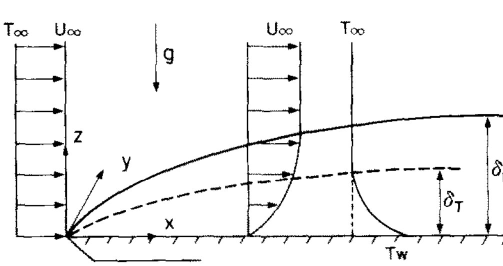

FIG. 1. Schematic diagram of the basic flow field.

These variables are of the same form as those used by Chen and Chen [4] except that in the pressure perturbation term a factor of $Pr^{1/3}$ is included instead of $Pr^{4/3}$. Thus, the dimensionless equations governing the disturbances can be written as

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0 \tag{4}$$

$$\frac{1}{Pr}\left(\bar{U}_b\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{u}\frac{\partial \bar{U}_b}{\partial \bar{x}} + \bar{W}_b\frac{\partial \bar{u}}{\partial \bar{z}} + \bar{w}\frac{\partial \bar{U}_b}{\partial \bar{z}}\right) = -\frac{1}{Re}\frac{1}{Pr^{2/3}}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{Re\,Pr^{2/3}}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} \tag{5}$$

$$\frac{1}{Pr}\left(\bar{U}_b\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{W}_b\frac{\partial \bar{v}}{\partial \bar{z}}\right) = -\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{1}{Re\,Pr^{2/3}}\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2} \tag{6}$$

$$\frac{1}{Pr}\left(\bar{U}_b\frac{\partial \bar{w}}{\partial \bar{x}} + \bar{u}\frac{\partial \bar{W}_b}{\partial \bar{x}} + \bar{W}_b\frac{\partial \bar{w}}{\partial \bar{z}} + \bar{w}\frac{\partial \bar{W}_b}{\partial \bar{z}}\right) = -\frac{\partial \bar{p}}{\partial \bar{z}} + Ra_0\,\bar{t} + \frac{1}{Re\,Pr^{2/3}}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{z}^2} \tag{7}$$

$$\bar{U}_b\frac{\partial \bar{t}}{\partial \bar{x}} + \bar{u}\frac{\partial \bar{\theta}_b}{\partial \bar{x}} + \bar{W}_b\frac{\partial \bar{t}}{\partial \bar{z}} + \bar{w}\frac{\partial \bar{\theta}_b}{\partial \bar{z}} = \frac{1}{Re\,Pr^{2/3}}\frac{\partial^2 \bar{t}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{t}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{t}}{\partial \bar{z}^2}. \tag{8}$$

Since our primary interest is in the effect of the x dependence of perturbation quantities on the onset of thermal instabilities, for the present study we restrict ourselves to the case of very large Prandtl numbers, as discussed in the second paragraph of the Introduction. Then all terms representing the inertia effect in the equations of motion are negligible [4, 5]. In addition, the terms involving $\partial\bar{p}/\partial\bar{x}$, $\partial^2\bar{u}/\partial\bar{x}^2$, $\partial^2\bar{v}/\partial\bar{x}^2$, $\partial^2\bar{w}/\partial\bar{x}^2$ and $\partial^2\bar{t}/\partial\bar{x}^2$ can also be omitted compared with other terms so that the resulting system of equations becomes

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0 \tag{9}$$

$$\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} = 0 \tag{10}$$

$$-\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2} = 0 \tag{11}$$

$$-\frac{\partial \bar{p}}{\partial \bar{z}} + Ra_0\,\bar{t} + \frac{\partial^2 \bar{w}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{w}}{\partial \bar{z}^2} = 0 \tag{12}$$

$$\bar{U}_b\frac{\partial \bar{t}}{\partial \bar{x}} + \bar{u}\frac{\partial \bar{\theta}_b}{\partial \bar{x}} + \bar{W}_b\frac{\partial \bar{t}}{\partial \bar{z}} + \bar{w}\frac{\partial \bar{\theta}_b}{\partial \bar{z}} = \frac{\partial^2 \bar{t}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{t}}{\partial \bar{z}^2}. \tag{13}$$

Manipulating $\partial^2(11)/\partial y\partial z - \partial^2(12)/\partial y^2$ and utilizing the continuity equation (9) in conjunction with equation (10), we obtain

$$\left(\frac{\partial^2}{\partial \bar{y}^2} + \frac{\partial^2}{\partial \bar{z}^2}\right)^2 \bar{w} + Ra_0\frac{\partial^2 \bar{t}}{\partial \bar{y}^2} = 0. \tag{14}$$

Next, we note from equation (10) and by considering boundary conditions at $\bar{z} = 0$ and ∞ that $\bar{u} \equiv 0$. Speaking more specifically, we note by comparing the right-hand sides of equations (5)–(7) that $\bar{u}$ is of lower order of magnitude than $\bar{v}$ and $\bar{w}$ as $Pr \to \infty$. This is consistent with the observation that in the limiting case of $Pr \to \infty$, the velocity perturbation in the streamwise direction disappears, as discussed in Chen and Chen [4] and references thereof. Hence equation (13) can now be reduced to

$$\bar{U}_b\frac{\partial \bar{t}}{\partial \bar{x}} + \bar{W}_b\frac{\partial \bar{t}}{\partial \bar{z}} + \bar{w}\frac{\partial \bar{\theta}_b}{\partial \bar{z}} = \frac{\partial^2 \bar{t}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{t}}{\partial \bar{z}^2}. \tag{15}$$

For the convenience of later analysis, equations (14) and (15) are rewritten in dimensional form as

$$\nu\nabla_2^2\nabla_2^2 w' + g\beta\frac{\partial^2 t'}{\partial y^2} = 0 \tag{16}$$

$$U_b\frac{\partial t'}{\partial x} + w'\frac{\partial T_b}{\partial z} + W_b\frac{\partial t'}{\partial z} = \kappa\nabla_2^2 t' \tag{17}$$

where

$$\nabla_2^2(\cdot) = \frac{\partial^2(\cdot)}{\partial y^2} + \frac{\partial^2(\cdot)}{\partial z^2}.$$

In predicting the onset of longitudinal vortex rolls, which are periodic in the y direction, the following form of normal modes for the perturbation quantities are employed:

$$w'(x,y,z) = w^*(x,z)\,e^{i\alpha y}$$
$$t'(x,y,z) = t^*(x,z)\,e^{i\alpha y}. \tag{18}$$

Substituting these expressions into equations (16) and (17), we obtain

$$\nu\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right)^2 w^* - g\beta\alpha^2 t^* = 0 \tag{19}$$

$$U_b\frac{\partial t^*}{\partial x} + W_b\frac{\partial t^*}{\partial z} + w^*\frac{\partial T_b}{\partial z} = \kappa\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right)t^*. \tag{20}$$

Similarity solutions for the amplitude functions $w^*(x,z)$ and $t^*(x,z)$ are sought by introducing a similarity variable based on $\delta_t(x)$ which is the thermal boundary layer thickness of the basic flow field

$$\zeta = z/\delta_t(x).$$

Accordingly, a new set of dimensionless variables are defined by using $\kappa/\delta_t(x)$ and Δt as reference velocity and temperature difference

$$\tilde{w}(\zeta) = \frac{\delta_t}{\kappa}w^*(x,z)$$

$$\tilde{t}(\zeta) = \frac{t^*(x,z)}{\Delta t}$$

$$\tilde{\alpha} = \alpha\delta_t$$

$$(\tilde{U}_b(\zeta), \tilde{W}_b(\zeta)) = \frac{\delta_t}{\kappa}(U_b, W_b)$$

$$\tilde{\theta}_b(\zeta) = \frac{T_b - T_\infty}{\Delta t}.$$

Table 1. Approximating polynomials used to represent the basic quantities in the integral method of boundary layer analysis and corresponding boundary layer thicknesses, where $\zeta_h = z/\delta_h$ and $\zeta = z/\delta_t$

U_b/U_∞	$(T_b - T_\infty)/\Delta t$	$\frac{\delta_h}{x}\sqrt{Re_x}$	δ_t/δ_h	$\frac{W_b}{U_\infty}\sqrt{Re_x}$
$\frac{3}{2}\zeta_h - \frac{1}{2}\zeta_h^3$	$1 - \frac{3}{2}\zeta + \frac{1}{2}\zeta^3$	4.64	$Pr^{-1/3}$	$1.74\left(\zeta_h^2 + \frac{1}{2}\zeta_h^4\right)$

Table 2. The values of parameter K for various cases

Case	Value of K
A1, B1	8.0736
A2, B2	16.1472
A3, B3	−8.0736
A4, B4	0

Then, equations (19) and (20) are nondimensionalized as follows:

$$(D^2 - \tilde{\alpha}^2)^2\tilde{w} - Ra\,\tilde{\alpha}^2\tilde{t} = 0 \qquad (21)$$

$$(D^2 - \tilde{\alpha}^2)\tilde{t} + \left(-\delta_t\frac{\partial\zeta}{\partial x}\tilde{U}_b - \tilde{W}_b\right)D\tilde{t} = \tilde{w}D\tilde{\theta}_b \qquad (22)$$

where $D(\cdot) = d(\cdot)/d\zeta$. The boundary conditions are $\tilde{w} = D\tilde{w} = \tilde{t} = 0$ at $\zeta = 0$ and ∞. Solving the above set of ordinary differential equations requires information on the basic flow field. Instead of using the exact solutions for the mixed convection flow problem elucidated in ref. [8], we adopt approximating polynomials which are used in the integral analysis of momentum and energy equations for the forced convection boundary layer flow problem. The fact that we are employing forced convection boundary layer flow can be justified if we recall that large Prandtl number fluids as considered in this study are less susceptible to buoyancy effects [9]. In Table 1, third-order polynomials for U_b and T_b and a fourth-order polynomial for W_b are given. A careful examination of these approximating polynomials shows that as $Pr \to \infty$ the coefficient of the second term of the left-hand side of equation (22) is reduced to a very simple form

$$-\delta_t\frac{\partial\zeta}{\partial x}\tilde{U}_b - \tilde{W}_b = \frac{1}{2}\frac{\delta_t}{x}\zeta\tilde{U}_b - \tilde{W}_b = K\zeta^2\,(K = \text{const.}) \qquad (23)$$

where the values of K for various cases are shown in Table 2. Substituting equation (23) into equation (22) and eliminating $\tilde{t}$ from equations (21) and (22), we obtain

$$(D^2 + K\zeta^2 D - \alpha^2)(D^2 - \alpha^2)^2 w = Ra\,\alpha^2 w D\theta_b \qquad (24)$$

where it is agreed that from here on tildes (~) over dimensionless quantities are omitted for convenience. Now, if we denote $w_i(\zeta) = w(\zeta)$ for $\zeta \le 1$ and $w_0(\zeta) = w(\zeta)$ for $\zeta > 1$, then, according to the definition of the thermal boundary layer, equation (24) can be written in a divided fashion as follows:

for $\zeta \le 1$

$$(D^2 + K\zeta^2 D - \alpha^2)(D^2 - \alpha^2)^2 w_i = Ra\,\alpha^2 w_i D\theta_b \qquad (25)$$

for $\zeta > 1$

$$(D^2 + K\zeta^2 D - \alpha^2)(D^2 - \alpha^2)^2 w_0 = 0 \qquad (26a)$$

with boundary conditions

$$w_i = Dw_i = (D^2 - \alpha^2)^2 w_i = 0 \quad \text{at } \zeta = 0 \qquad (27)$$

$$D^n w_i = D^n w_0 (n = 0, 1, 2, 3, 4, 5) \quad \text{at } \zeta = 1 \qquad (28a)$$

$$w_0 = Dw_0 = (D^2 - \alpha^2)^2 w_0 = 0 \quad \text{as } \zeta \to \infty. \qquad (29a)$$

If we assume that the bottling effect whereby the temperature disturbance is contained within the thermal boundary layer of the basic flow [3] holds, then for $\zeta > 1$ we can take $t = 0$. Actually, this idea was proven to be valid by Choi [6] and Davis and Choi [7] who reported that in a liquid-film flow of a large Prandtl number fluid the temperature disturbances are confined within the thermal boundary layer at the onset of thermal instability. The stability equation can then be written as follows:

for $\zeta \le 1$

$$(D^2 + K\zeta^2 D - \alpha^2)(D^2 - \alpha^2)^2 w_i = Ra\,\alpha^2 w_i D\theta_b \qquad (25)$$

for $\zeta > 1$

$$(D^2 - \alpha^2)^2 w_0 = 0 \qquad (26b)$$

with boundary conditions

$$w_i = Dw_i = (D^2 - \alpha^2)^2 w_i = 0 \quad \text{at } \zeta = 0 \qquad (27)$$

$$D^n w_i = D^n w_0 (n = 0, 1, 2, 3, 4) \quad \text{at } \zeta = 1 \qquad (28b)$$

$$w_0 = Dw_0 = 0 \quad \text{as } \zeta \to \infty. \qquad (29b)$$

For each of the problem sets (25)–(29a) and (25)–(29b), we can additionally study the effect of parallel basic flow ($W_b \equiv 0$ in equation (17)), the effect of perturbation quantities which are independent of $x(\partial t'/\partial x = 0$ in equation (17)) and the combined effect of both. Thus, stability analyses are carried out for eight different subproblems altogether. The values of K appearing in equations (23)–(26) differ according to the subproblems considered and are listed in Table 2, where symbol 'A' corresponds to problem set (25)–(29a) and 'B' to (25)–(29b), while symbol '1' refers to the subproblem with ($W_b \neq 0$, $\partial t'/\partial x \neq 0$), '2' with ($W_b = 0$, $\partial t'/\partial x \neq 0$), '3' with ($W_b \neq 0$, $\partial t'/\partial x = 0$) and '4' with ($W_b = 0$, $\partial t'/\partial x = 0$).

3. METHOD OF SOLUTION

Since the coefficients of the ordinary differential equation (25) are polynomials in $\zeta(\le 1)$, we can use the well-known power series solution method [10] and construct a general solution for w_i in the form

$$w_i = \sum_{j=0}^{5} C_j f_j(\zeta)$$

where $C_j(j = 0, 1, 2, 3, 4, 5)$ are arbitrary constants and $f_j(\zeta)$ are rapidly convergent power series

$$f_j(\zeta) = \sum_{n=0}^{\infty} b_n^{(j)}(Ra, \alpha)\zeta^n \Big| (j = 0, 1, 2, 3, 4, 5).$$

The series coefficients for $n \leq 5$ are specified as

$$b_{-3}^{(j)} = b_{-2}^{(j)} = b_{-1}^{(j)} = 0$$
$$b_n^{(j)} = \delta_{nj}$$

and those for $n \geq 6$ are determined in terms of the preceding coefficients obeying the recurrence formulas generated from equation (25)

$$b_n^{(j)} = \Big[3\alpha^2(n-2)(n-3)(n-4)(n-5)b_{n-2}^{(j)} - K(n-3)(n-4)(n-5)(n-6)(n-7)b_{n-3}^{(j)} - 3\alpha^4(n-4)(n-5)b_{n-4}^{(j)} + 2K\alpha^2(n-5)(n-6)(n-7)b_{n-5}^{(j)} + \left(\alpha^2 + \frac{3}{2}\alpha^2 Ra\right)b_{n-6}^{(j)} - \alpha^4 K(n-7)b_{n-7}^{(j)} + \frac{3}{2}\alpha^2 Ra\, b_{n-8}^{(j)}\Big] \Big/ \{n(n-1)(n-2)(n-3)(n-4)(n-5)\}.$$

The constants $C_j(j = 0, 1, 2, 3, 4, 5)$ are chosen to satisfy the boundary and interface conditions. From the boundary conditions at $\zeta = 0$, we obtain

$$C_0 = C_1 = 0, \qquad C_4 = \frac{\alpha^2}{6}C_2. \tag{30}$$

Thus, w_i can now be written as

$$w_i = C_2\left(f_2 + \frac{\alpha^2}{6}f_4\right) + C_3 f_3 + C_5 f_5. \tag{31}$$

As for the solution of w_0, we first consider the case when the bottling effect of the thermal disturbances is not taken into account by writing equation (26a) for $\zeta > 1$ in the following form:

$$(D^2 + K\zeta^2 D - \alpha^2)Y = 0 \tag{32}$$

$$(D^2 - \alpha^2)^2 w_0 = Y. \tag{33}$$

We then begin by finding a solution which satisfies the third condition of equation (29a), that is

$$Y \to 0 \qquad \text{as } \zeta \to \infty.$$

Thus, the solution of equation (32) can be obtained through the WKB method [11]

$$Y \sim \exp\left[-\frac{K}{6}\zeta^3 - \int_1^{\zeta}\sqrt{\left(\frac{K^2}{4}\zeta^4 + K\zeta + \alpha^2\right)}\mathrm{d}\zeta\right] \Big/ \left(\frac{K^2}{4}\zeta^4 + K\zeta + \alpha^2\right)^{1/4}.$$

From this expression we can calculate $Y(1)$ and $Y'(1)$, which are exactly coincident with the values obtained through numerical integration from the asymptotic solution of equation (32) as $\zeta \to \infty$. Now it can be said that one of the boundary conditions as $\zeta \to \infty$ is replaced with two initial conditions at $\zeta = 1$.

Therefore, introducing the transformation of variables $s = \zeta - 1$, solving equation (32) near $s = 0$ (i.e. for $s < 1$) by a power series solution method and sequently solving equation (33) by an operator technique, we obtain near $s = \zeta - 1 = 0$

$$w_0 = C_6[e^{\alpha s}g(s) - e^{-\alpha s}h(s)]/4\alpha^2 + C_7 e^{-\alpha s} + C_8 s e^{-\alpha s} \tag{34a}$$

where functions $g(s)$ and $h(s)$ are power series as shown below

$$g(s) = \sum_{n=0}^{\infty}\frac{g_n}{(n+1)(n+2)}s^{n+2} - \frac{1}{\alpha}\sum_{n=0}^{\infty}\frac{g_n}{n+1}s^{n+1}$$
$$g_n = -[(K+2\alpha)(n-1)g_{n-1} + K(2n-4+\alpha)g_{n-2} + K(n-3+2\alpha)g_{n-3} + K\alpha g_{n-4}]/n(n-1)$$
$$g_{-1} = g_{-2} = 0$$
$$g_0 = Y(1)$$
$$g_1 = Y'(1) - \alpha Y(1)$$
$$h(s) = \sum_{n=0}^{\infty}\frac{h_n}{(n+1)(n+2)}s^{n+2} + \frac{1}{\alpha}\sum_{n=0}^{\infty}\frac{h_n}{n+1}s^{n+1}$$
$$h_n = -[(K-2\alpha)(n-1)h_{n-1} + K(2n-4-\alpha)h_{n-2} + K(n-3-2\alpha)h_{n-3} - K\alpha h_{n-4}]/n(n-1)$$
$$h_{-1} = h_{-2} = 0$$
$$h_0 = Y(1)$$
$$h_1 = Y'(1) + \alpha Y(1).$$

Finally, application of boundary conditions (28a) with (31) and (34a) results in the following secular equation of a 6 × 6 determinant:

$$\begin{vmatrix} 0 & 1 & 0 & -(f_2 + \alpha^2/6f_4) & -f_3 & -f_5 \\ 0 & -\alpha & 1 & -(f_2' + \alpha^2/6f_4') & -f_3' & -f_5' \\ 0 & \alpha^2 & -2\alpha & -(f_2'' + \alpha^2/6f_4'') & -f_3'' & -f_5'' \\ 0 & -\alpha^3 & 3\alpha^2 & -(f_2''' + \alpha^2/6f_4''') & -f_3''' & -f_5''' \\ Y & \alpha^4 & -4\alpha^3 & -(f_2^{\text{iv}} + \alpha^2/6f_4^{\text{iv}}) & -f_3^{\text{iv}} & -f_5^{\text{iv}} \\ Y' & -\alpha^5 & 5\alpha^4 & -(f_2^{\text{v}} + \alpha^2/6f_4^{\text{v}}) & -f_3^{\text{v}} & -f_5^{\text{v}} \end{vmatrix}_{\text{at } \zeta=1} = 0 \tag{35}$$

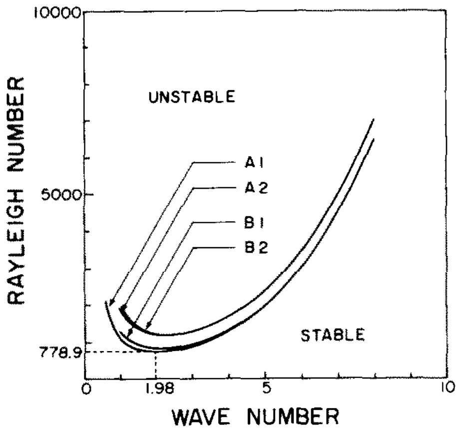

FIG. 2. Neutral stability curves for various cases.

Table 3. Critical values obtained by the present analysis under various assumptions

Case	$\tilde{\alpha}_c$	Ra_c	$Gr_x/Re_x^{1.5}$
A1	1.98	778.9	7.797
A2	2.23	1194.5	11.957
B1	2.08	872.7	8.736
B2	2.26	1231.1	12.324
B3†	1.76	438	4.38
B4†	1.88	584	5.85

†These cases have been treated separately in ref. [12].

where primes denote differentiation with respect to ζ.

We next consider the case when the bottling effect is taken into account. In this case, an exact solution of equation (26b) satisfying boundary conditions (29b) can be obtained in a very simple form near $s = \zeta - 1 = 0$

$$w_0 = C_6 e^{-\alpha s} + C_7 s e^{-\alpha s} \tag{34b}$$

where $s = \zeta - 1$ as before. By applying boundary conditions (28b) with (31) and (34b), we obtain the secular equation of a 5×5 determinant which consists of the elements inside the dotted line in equation (35).

Equation (35) is an eigenvalue problem, through which the value of Ra can be found as an eigenvalue for a given value of α. Once the neutral stability curve, that is, the curve of α vs Ra is obtained, the minimum value of Ra and the corresponding wave number α can be determined which marks the onset of thermal instability.

4. RESULTS AND DISCUSSION

The neutral stability curves for four cases where the x-derivatives of the perturbation quantities are assumed are shown in Fig. 2. The critical Rayleigh numbers Ra_c and the corresponding critical wave numbers α_c obtained from the neutral stability curves in Fig. 2 are tabulated in Table 3 together with the critical buoyancy parameters $Gr_x/Re_x^{1.5}$ which are proportional to the critical Rayleigh number, Ra_c. Case A1 is most important, where non-parallel basic flow and x-dependent perturbation quantities are considered. The results for other cases are listed to study qualitatively the effect of parallel basic flow and the bottling effect of temperature perturbation. In Table 4, the result for case A1 is also compared with previous theoretical and experimental works performed under different situations. The effect of Reynolds number on the critical Grashof number may be studied separately from Fig. 3.

To include Wu and Cheng's [1] data may be insignificant because it has already been pointed out in a few previous works such as Chen and Mucoglu [14] and Moutsoglou *et al.* [2] that a couple of numerical errors were involved in their work. The vortex instability of forced convection flow over a horizontal flat plate was also treated by Moutsoglou *et al.* [2] as a part of their thorough investigation on vortex instability of mixed convection flow. Like Wu and Cheng, they simply followed Haaland and Sparrow [3] by adopting the form of perturbation quantities which are independent of x. As shown in Table 4 and Fig. 3, their theoretical values of the parameter $Gr_x/Re_x^{1.5}$ are about two orders of magnitude lower than Gilpin *et al.*'s [13] experimental values. This may be acceptable when we consider that instability must grow for a finite disturbance before its amplitude is large enough to be observed [2, 4, 13]. However, questions can be raised about the values of Moutsoglou *et al.*'s [2] critical wave numbers which are either zero for forced convection flow or extremely small for mixed convection flow in the light of Gilpin *et al.*'s [13] experimental values. In fact, it is quite curious why Moutsoglou *et al.* did not make any comments on their unrealistically small values of their critical wave numbers. An answer to these doubts can be found by considering x-dependent perturbation quantities, because the concept of x-dependent perturbation quantities which used to be valid for natural convection basic flow may fail to be so for forced convection basic flow, where stronger transport of disturbance energy along the streamwise direction may stabilize the flow. In fact, Chen and Chen [4] already argued that large Reynolds number may suppress the occurrence of thermal instability in the forced convection problem, although they did not specifically mention the role of x-dependent perturbation quantities.

On the other hand, it is remarkable that the present result for Case A1 agrees exactly with Chen and Chen's [4] result for their $m = 0$ (flat plate) when the following conversion is made between the two:

$$K^+ = K_0^+[2x^+]^{1/2}$$

$$= \frac{K_0 L_0}{Re^{1/2} Pr^{1/3}} \left[\frac{2x}{L_0}\right]^{1/2} \Rightarrow \frac{\sqrt{2}}{4.64} \alpha \delta_t = \frac{\sqrt{2}}{4.64} \tilde{\alpha}$$

$$Ra^+ = Ra_0^+[2x^+]^{3/2}$$

$$= \frac{g\beta\Delta t (L_0 Re^{-1/2} Pr^{-1/3})^3}{\alpha \nu} \left(\frac{2x}{L_0}\right)^{3/2} \Rightarrow \left(\frac{\sqrt{2}}{4.64}\right)^3 Ra.$$

Table 4. Comparison of critical values with those obtained by previous works on vortex instability†

	Basic flow	Perturbation quantities	Pr	$\tilde{\alpha}_c$	$Gr_x/Re_x^{1.5}$
Wu and Cheng [1]	forced convection	$\frac{\partial(.)}{\partial x} = 0$	10	3.70 ($a^* = 1.72$)	75.48
			10^2	2.95 ($a^* = 2.95$)	13.46
			10^3	1.81 ($a^* = 3.90$)	2.406
			10^4	1.55 ($a^* = 7.2$)	1.806
Moutsoglou *et al.* [2]	forced convection	$\frac{\partial(.)}{\partial x} = 0$	0.7	0	—
			7	0	—
	mixed convection	$\frac{\partial(.)}{\partial x} = 0$	0.7	0.005–0.314 ($\alpha^* = 0.001$–0.06)	0.447
			7	0.0019–0.087 ($\alpha^* = 0.0008$–0.036)	0.434
Chen and Chen [4] ($m = 0.0$)	forced convection	$\frac{\partial(.)}{\partial x} \neq 0$	1–∞	1.97 ($K^+ = 0.6$)	7.78–8.13 ($Ra^+ = 22$–23)
Present analysis	forced convection	$\frac{\partial(.)}{\partial x} \neq 0$	∞	1.98	7.797
Gilpin *et al.* [13]	experiment		6–10 (water)	2.17–4.97 ($x_c/\lambda = (0.16 - 0.31)Re_x^{1/2}$)	46–110

† Inside the parentheses are the numbers as expressed in the original papers.

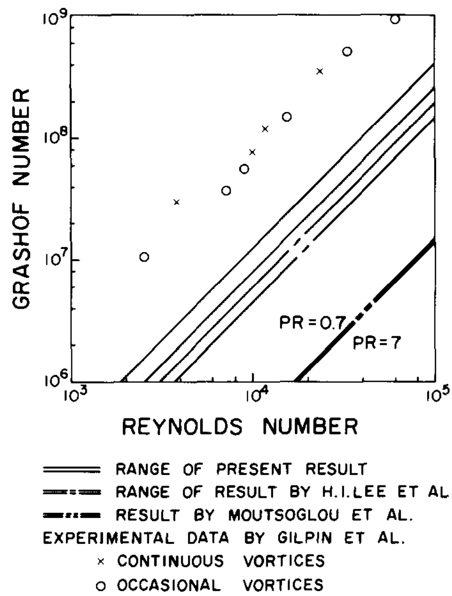

FIG. 3. Local Reynolds number vs critical local Grashof number marking the onset of longitudinal vortex rolls.

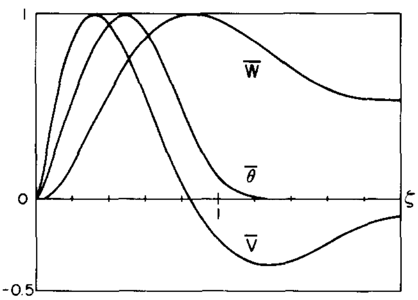

FIG. 4. Distributions of the normalized disturbances at the onset of longitudinal vortices for Case A1.

To be more precise, we find by substituting $\tilde{\alpha}_c = 1.98$ and $Ra_c = 778.9$ in the above expressions that $K^+_{\text{critical}} \simeq 0.6$ and $Ra^+_{\text{critical}} \simeq 22$, which are exactly coincident with their Fig. 3. It is also important to note that their critical Rayleigh number and critical wave number depend on Prandtl number only very weakly, which justifies our simplification made below equation (8) and above equation (15). Since the present values of the Rayleigh number and wave number at the critical point in Blasius' flow are exactly coincident with those obtained by Chen and Chen [4], it is unnecessary to repeat them in elaborating that the predicted wave number agrees well with the experimental observations of Gilpin *et al.* [13] and that the calculated value of the parameter $Gr_x/Re_x^{1.5}$ is about one order of magnitude lower than the experimental results, which is usually the case with the thermal instability problems. However, we add that from Fig. 6 of Gilpin *et al.* the agreement between the critical wave numbers is better for smaller temperature differences, i.e. for stronger forced convection basic flow situations.

Now that the validity of the present method has been established, its two-fold advantages can be considered. Firstly, it requires less computer time and memory. Secondly and more importantly, it provides useful information for further studies on the effect of the terms W_b and $\partial t'/\partial x$, and thus on the onset mechanism of thermal instabilities. For instance, let us consider the fact that there exist no true solutions for Cases A3 and A4. For Case A3, where $W_b \neq 0$ and $\partial t'/\partial x = 0$, the value of K is negative as can be seen from Table 2. Therefore, no solution of equation (32) exists which satisfies the condition that $Y \to 0$ as $\zeta \to \infty$. For Case A4, where $W_b = 0$ and $\partial t'/\partial x = 0$, the critical wave number $\tilde{\alpha}_c$ becomes zero, contradicting the experimental observations. Therefore, it is strongly recommended that the conventional analysis on the thermal instability of forced (or mixed) convection flow in which the perturbation quantities are assumed to be independent of the streamwise coordinate should be re-examined.

The distributions of perturbation quantities at the onset of thermal instability for Case A1 are illustrated in Fig. 4. Here, each of the perturbation amplitudes is normalized with respect to its maximum value since it can be calculated only up to multiplicative constants. It is of great interest to observe that the temperature disturbance is confined mainly in the thermal boundary layer, which is a strong indication that the bottling effect of the temperature disturbance may be a reasonable assumption. However, in utilizing this bottling effect in the stability problems, one has to be very careful about the possible occurrence of a discontinuous derivative of the temperature perturbation at the edge of the thermal boundary layer, as can be seen from equations (21) and (28b). Hence, the bottling effect can be considered only under such restrictions.

When the values of Ra_c and $Gr_x/Re_x^{1.5}$ for non-parallel and parallel basic flow models are compared, it can be seen that the former (Cases A1 and B1) represent lower values than the latter (Cases A2 and B2). This is considered to be due to the blowing effect of the basic transverse velocity component W_b. In other words, in the case of the non-parallel basic flow model, the thermal boundary layer is more susceptible to instabilities due to the presence of W_b. Lastly, if it is allowed to proceed along with the afore-mentioned reservations on the bottling effect, then x-dependent disturbances (Cases B1 and B2) can be said to give more stable values than x-independent disturbances (Cases B3 and B4). This is due to the presence of the x transport term in equation (17), namely, $U_b(\partial t'/\partial x)$, which transfers the unstabilizing temperature disturbances t' along the flow direction so that the flow becomes stabilized.

5. CONCLUSION

The onset of thermal instability in the form of longitudinal vortex rolls occurring in the laminar flow over a horizontal flat plate heated isothermally from below has been examined by linear stability analysis for the limiting case of very large Prandtl numbers, under the assumption that the amplitudes of the perturbation quantities may have non-zero streamwise derivatives.

By introducing a similarity variable based on varying thermal boundary layer thickness and by employing the non-parallel and parallel basic flow models in the form of approximating polynomials used in the integral method of forced convection boundary layer flow problem, the perturbation amplitudes can be sought in the convenient form of a fast convergent power series. Then, the neutral stability curves can be obtained by solving a secular equation generated from the interface conditions at the edge of the thermal boundary layer.

The critical Rayleigh number and the critical wave number are shown to agree exactly with the theoretical values of Chen and Chen [4], explaining very well the experimental results of Gilpin *et al.* [13]. More reasonable interpretations of other previous studies are given with the emphasis that the x-dependent perturbation quantities must be considered in the case of forced convection basic flow, which acts to stabilize the flow. The critical buoyancy parameter for non-parallel basic flow is smaller than that for parallel basic flow, which verifies the blowing effect of the basic transverse velocity component. Although the bottling effect of temperature perturbation may be a useful tool to simplify the formulation of the stability problem, appropriate caution should be taken when it is adopted.

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ANALYSE DE L'INSTABILITE THERMIQUE DE L'ECOULEMENT FORCE DE CONVECTION SUR UNE PLAQUE PLANE HORIZONTALE ET ISOTHERME

Résumé—L'instabilité thermique de la convection forcée sur une plaque plane horizontale isotherme, en forme de tourbillons longitudinaux est étudiée en introduisant une dépendance spatiale tridimensionnelle des quantités de perturbations. Le système d'équations de stabilité est fortement simplifié en considérant le cas limite des très grands nombres de Prandtl et en recherchant des solutions de similitude pour les fonctions amplitude de quantités de perturbation. L'effet de la perturbation de température dépendante de x est montré stabiliser l'écoulement en comparaison de x, ce qui explique très bien les résultats thermiques et les observations expérimentales.

UNTERSUCHUNG DER THERMISCHEN INSTABILITÄT EINER ERZWUNGENEN KONVEKTIONSSTRÖMUNG ÜBER EINE ISOTHERME HORIZONTALE EBENE PLATTE

Zusammenfassung—Es wurde die thermische Instabilität einer erzwungenen Konvektionsströmung über eine isotherme horizontale ebene Platte in Form von Longitudinalwirbeln durch Einführung der dreidimensional räumlichen Abhängigkeit der Störgrößen untersucht. Das System der Stabilitätsgleichungen wurde wesentlich vereinfacht durch Betrachten des Grenzfalls sehr großer Prandtl-Zahlen und durch Aufsuchen von Ähnlichkeitslösungen für die Amplitudenfunktion der Störgrößen. Es zeigt sich, daß die x-abhängigen Temperaturstörungen im Gegensatz zu den x-unabhängigen Temperaturstörungen die Strömung stabilisieren. Dies erklärt sehr gut die vorhandenen theoretischen Ergebnisse und die experimentellen Beobachtungen.

ИССЛЕДОВАНИЕ ТЕПЛОВОЙ УСТОЙЧИВОСТИ ВЫНУЖДЕННОГО КОНВЕКТИВНОГО ТЕЧЕНИЯ НАД ИЗОТЕРМИЧЕСКОЙ ГОРИЗОНТАЛЬНОЙ ПЛОСКОЙ ПЛАСТИНОЙ

Аннотация—Тепловая устойчивость вынужденного конвективного течения относительно возмущений типа продольных вихрей над изотермической горизонтальной плоской пластиной исследуется методом возмущений, зависящих от трех пространственных координат. Система уравнений устойчивости существенно упрощается за счет рассмотрения предельного случая очень больших чисел Прандтля и отыскания автомодельных решений для амплитудных функций величин возмущений. Показано, что влияние возмущений температуры, зависящих от x, стабилизирует течение по сравнению с возмущениями температуры, не зависящими от x, что хорошо объясняет имеющиеся теоретические и экспериментальные результаты.